

STATISTICAL MODELLING OF ANNUAL MAXIMUM RAINFALL FOR BOTSWANA USING EXTREME VALUE THEORY

Wilson Moseki Thupeng

Department of Statistics, University of Botswana, Private Bag UB 00705, Gaborone, Botswana

ABSTRACT

The objective of this paper is to find the best-fit probability model for annual maximum rainfall data for Botswana for the period January 1901 to December 2012. The Kwiatkowski-Philips-Schmidt-Shin test of stationarity on the data shows that the time series is stationary. Based on visual inspection of the generalized quantile plot, three extreme value probability distributions are considered, belonging to the Gumbel and Frechet maximum domains of attraction: the gamma, lognormal and BurrXII. The maximum likelihood method is used to estimate the parameters of each distribution under study. The empirical CDF plots and Q-Q plots of the data show that the three models are highly competitive in terms of goodness-of-fit. Formal model assessment criteria, namely, the Anderson-Darling test and the Bayesian Information Criterion agree on the ranking of the three models: the lognormal distribution gives the best fit to the data followed closely by the gamma whilst the BurrXII distribution comes a distant third.

KEYWORDS: Annual Maximum Rainfall, BurrXII, Domain of Attraction, Extreme Value Theory, Gamma, Lognormal

AMS Classification: 62G32

Article History

Received: 29 Dec2018 | Revised: 10 Jan 2019 | Accepted: 21 Jan 2019

INTRODUCTION

Precipitation and temperature extremes are considered to be the most important climate events and have been extensively explored over the past several decades, according to [1] and [2]. Water is one of the most important natural resources with diverse uses in the agricultural, domestic and industrial sectors. Low or excessive rainfall can lead to water scarcity or floods, which may cause great damage to human health, the environment, and property, leading to massive social and economic losses. While nearly a third of the people in Africa already live in the drought-prone areas, it is estimated that climate change will add up to over 80 million people at risk of hunger by 2080 [3-4]. The debilitating effects of climate change have not spared Botswana, which is characterized by a semi-arid climate with an erratic and low rainfall (450 mm per annum), high evaporation (more than 2000 mm per annum) and limited fresh water supplies. According to [5], typically, between 1965 and 2006 the country has suffered from several droughts and flood-related disasters which have affected 1.5 million people and caused economic losses of over US\$ 3 million to the country. Most of the water used for domestic, industrial and agricultural purposes in Botswana is from dams. Consequently, water demand far exceeds water supply as the dams are drying up due to erratic and low rainfall coupled with high evaporation rates across the country. This situation is a major threat to the livelihoods of the country's population, especially the rural stratum.

According to [6], about 70% of the rural households derive their livelihoods from agriculture through subsistence farming and crop production is mainly based on rain-fed farming.

The main motivation for statistical modelling of weather extremes is reliability: the ability of a system or component to withstand the weather conditions for the whole of its working life. For instance, for civil engineers to determine how high the wall of a reservoir dam should be, they need to estimate what the heaviest (maximum) rainfall amount will be over some specified future time period. This can only be achieved by using historical data and fitting an appropriate statistical model for them. That is, a model for risk is developed by selecting a probability distribution that gives the best fit to the data at hand. Statistical models for extreme values are based on an asymptotic theory called extreme value theory. Extreme value theory provides the statistical framework to make inferences about the probability of very rare and extreme events [7]. The theory is based on the probability distribution of the unusually low or large values of a time series.

The application of extreme value theory to the analysis and modelling of extreme rainfall from different regions of the world bears testimony to the importance of determining the frequency of extreme rainfall events in different parts of the world. For instance, [8] determine the best-fit models for the frequency analysis of extreme monthly rainfall events in Bangladesh and estimate extreme quintiles for several periods of time; [7] provide the first application of extreme value distributions to model minimum annual rainfall in Zimbabwe; [4] model a long-term annual rainfall data from 1961 to 2003 at 11 synoptic stations across Botswana using extreme value distributions. [9] apply extreme value theory to model rainfall data from South Korea; [10] use extreme value theory to model rainfall data from Europe and the USA; [11] apply extreme value theory to Greece's rainfall data; while [12] model rainfall data from Italy using extreme value distributions.

Though modelling of extreme rainfall has been shown to be important, not much research in this area has been done in Botswana. Perhaps the most significant research so far known to the author is work by [4], which involves a regional frequency analysis of annual rainfall data of covering 40 years at 11 synoptic stations across Botswana using extreme value distributions. The results of the study show that all the stations behaved homogeneously and followed the generalized extreme value (GEV) distribution. However, as pointed out by [13], rainfall should be monitored overtime as significant changes in the rainfall may lead to floods or drought. Knowledge of the distribution, intensity, and duration of precipitation extremes is important to stakeholders like climate change researchers, businesses, governments and members of the general public as forecasting of future events may help them adequately prepare for any extreme events.

The main objective of the present study is to find the best-fit model for determining the frequency of extreme rainfall events in Botswana by using historical annual maximum rainfall data from 1901 to 2012. The other objective seeks to determine whether there has been any monotonic change (trend) in the annual maximum average rainfall in Botswana in the period 1901-2012. We compare the performance of three different distributions: the gamma and lognormal selected from the Gumbel and, the BurrXII from the Frechet maximum domains of attraction of normalized maxima.. The remainder of the paper is organized as follows. The second section provides a description of the annual maximum rainfall data in Botswana from 1901 to 2012 and the extreme value methodology employed in the study. The third section presents the results and a discussion of the study. The fourth section contains the conclusions of the study.

DATA AND RESEARCH METHODOLOGY

Data Description

The data used for this study consist of annual maximum rainfall for Botswana extracted from the mean historical monthly rainfall for Botswana during the time period 1901-2012. The dataset was produced by the Climatic Research Unit (CRU) of the University of East Anglia (UEA), United Kingdom, and is available at the World Bank website http://sdwebx.worldbank.org/climateportal/index.cfm?page=downscaled_data_download&menu=historical. The choice of the time period is based on the quality and availability of data. The data produced by the CRU are of very high quality with no missing values.

Theoretical Foundation of Extreme Value Theory

The basis for extreme value theory is the Fisher-Tippet-Gnedenko Theorem (Extremal Types Theorem), first discovered by [14] and later proved in full generality by [15]. The theorem postulates the asymptotic distribution of extreme order statistics. It states that if X_1, X_2, \dots, X_n is a sample of independent and identically distributed (iid) random variables with a common distribution function F , and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ is the increasingly ordered sample, and $X_{(n)} = \max(X_1, X_2, \dots, X_n)$ is normalized with constants $a_n > 0$ (for all n) and b_n such that

$$\lim_{n \rightarrow \infty} P\left(\frac{X_{(n)} - b_n}{a_n} \leq z\right) = \lim_{n \rightarrow \infty} \{F(a_n z + b_n)\}^n = G_\xi(z) \text{ for all } z \in \mathbb{R} \tag{2.1}$$

where $G_\xi(z)$ is a non-degenerate distribution function, then $G_\xi(z)$ is called an extreme value distribution function. It is given by Equation (2.2).

$$G_\xi(z) = \begin{cases} \exp\left(- (1 + \xi z)^{-\frac{1}{\xi}}\right), & 1 + \xi z > 0 \text{ if } \xi \neq 0 \\ \exp(-\exp(z)), & z \in \mathbb{R} \text{ if } \xi = 0 \end{cases} \tag{2.2}$$

where ξ is called the extreme value index (EVI), a parameter that determines the heaviness of the tail of the distribution. And, the only possible types for $G_\xi(z)$ (limiting distributions for the linearly normalized variable $Z = \frac{X_{(n)} - b_n}{a_n}$) as defined

in Equation (2.1) are the Gumbel (which has an infinite right endpoint with the tail that decays much faster than the tail of Frechet) with distribution function $G_\xi(z) = \exp\left(-\exp\left(\frac{z-b}{a}\right)\right), z \in \mathbb{R}$; Frechet (with an infinite right endpoint) having the distribution function $G_\xi(z) = \begin{cases} 0, & z \leq b \\ \exp\left(-\left(\frac{z-b}{a}\right)^{-\alpha}\right), & z > b \end{cases}$, and the Weibull (short-tailed with a finite right endpoint) with

$$G_\xi(z) = \begin{cases} \exp\left(-\left[-\left(\frac{z-b}{a}\right)\right]^\alpha\right), & z < b \\ 1, & z \geq b \end{cases}$$

Important to note is that all the three types of distributions have a location and scale parameters, b and a , with $a > 0$, respectively. In addition, the Frechet and Weibull have a positive shape parameter α , which gives the EVI through the relations $\xi = 1/\alpha$ and $\xi = -1/\alpha$ for the former and latter, respectively.

A reformulation of these three limiting distributions for the normalized maximum M_n involves combining them to form a single family called the generalized extreme value (GEV) distribution with a cumulative distribution function given by equation (2.3).

$$G(z; \mu, \sigma, \xi) = \begin{cases} \exp\left(-\left[1 + \xi\left(\frac{z-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right), & \left\{z: 1 + \xi\left(\frac{z-\mu}{\sigma}\right) > 0\right\}, \xi \neq 0 \\ \exp\left(-\exp\left[-\left(\frac{z-\mu}{\sigma}\right)\right]\right), & z \in \mathbb{R}, \xi = 0 \end{cases} \quad (2.3)$$

with $-\infty < \mu < \infty$ and $\sigma > 0$ and $-\infty < \xi < \infty$. The μ , σ and ξ are the locations, scale and shape parameters, respectively, with ξ determining the rate of tail decay. The GEV distribution incorporates light-tailed Gumbel for $\xi \rightarrow 0$, heavy-tailed Frechet for $\xi > 0$, and short-tailed Weibull for $\xi < 0$. For details on introduction to statistical modelling of extremes, the reader is referred to [16].

Selecting the Maximum Domain of Attraction

A distribution function F is said to be in the domain of attraction for maxima of G_ξ , written $F \in D(G_\xi)$, whenever condition (2.1) holds. And, as pointed out by [17], testing whether $F \in D(G_\xi)$, for a certain ξ , is a crucial topic when an application of extreme value procedures is to be considered. However, in practice, the analyst has no information on the sign of the extreme value index, ξ , a priori. A brief overview of some testing procedures for the extreme value condition and for the statistical choice of the tail are discussed in [17]. However, such procedures are not easy to implement and, in practical situations, exploratory graphical tools like the quantile (Q-Q) plot and the mean excess plot are used to determine the tail behavior of the underlying distribution of sample data.

In this paper, for ease of implementation, we use a graphical tool called the generalized quantile plot defined by [18]. Given a sample of iid random variables X_1, X_2, \dots, X_n , with $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ the corresponding increasingly ordered sample, the generalized quantile plot is based on the generalized Hill estimator for ξ , which is a generalisation of the ordinary estimator of the slope of the k largest points of the Pareto Q-Q plot. From [20], the generalized Q-Q plot is a graph of the points $\left(\log\left(\frac{n+1}{k+1}\right), \log(X_{n-k,n}H_{k,n})\right)$, $k = 1, 2, \dots, n-1$, where $X_{n-k,n}H_{k,n}$, called the UH scores are usually written as $UH_{j,n} = X_{n-j,n}H_{j,n}$, $j = 1, 2, \dots, n-1$, and $H_{k,n}$ are the Hill estimates given by $H_{k,n} = 1/k \sum_{j=1}^k \log X_{n-j+1,n} - \log UH_{n-k,n}$.

Clearly, as pointed out by [18], for small k the generalized Q-Q plot becomes ultimately linear. To be noted also is that the generalized Hill estimator discussed in [18] was proposed earlier as an estimator for the extreme value index ξ using the slope of the generalized Q-Q plot in the case $\xi \in \mathbb{R}$. For these reasons, the slope of the generalized Q-Q plot can be used as a simple practical tool to check for the maximum domain of attraction condition for all the three types of extreme value distributions: the slope of the generalized Q-Q plot in the largest observations is positive if $\xi > 0$, indicating that the underlying distribution is heavy-tailed (in the Frechet domain); is fairly constant if $\xi = 0$, indicating that the underlying distribution is medium-tailed (in the Gumbel domain) and is negative if $\xi < 0$, indicating that the underlying distribution is short-tailed with a finite right endpoint (in the Weibull domain).

Testing for Stationarity

The statistical basis for parameter estimation and time series forecasting depends on the series being second-order (covariance) stationary: data not having any upward or downward trend or seasonal effects and, the mean or variance being consistent over time. Violation of stationarity in climate extremes can lead to wrong estimates of return levels derived from a given extreme value model [2]. There are many tests for stationarity in the statistical literature. In this study, we use the Kwiatkowski, Phillips, Schmidt and Shin test for stationarity introduced by in 1992.

The KPSS test tests the null hypothesis that a time series is stationary (has no unit root) against the alternative that the series is non-stationary due to the presence of a unit root. Details of the test are summarised in [2]. Briefly, the test starts with a specification of the model

$$x_t = \beta' D_t + \mu_t + u_t \text{ with } \mu_t = \mu_{t-1} + \varepsilon_t \text{ and } \varepsilon_t \sim WN(0, \sigma_t^2),$$

where β is a vector of regression coefficients, D_t contains deterministic components (constant or constant plus time trend), μ_t is a pure random walk with innovation variance σ_t^2 , and μ_t is a random error term of x_t , and may be heteroskedastic, ε_t denotes the error term of μ_t and is assumed to be a series of independent and identically distributed random variables with a mean of zero and constant variance σ_t^2 . The quantity t is the time index for each function, with $t = 1, 2, \dots, T$.

The test statistic for the KPSS test is a score statistic for testing $\sigma_t^2 = 0$ against the alternative that $\sigma_t^2 > 0$ and is defined by $K = \frac{1}{T^2} \sum_{t=1}^T \hat{S}_t^2 / \hat{\lambda}^2$ where $\hat{S}^2 = \sum_{j=1}^t \hat{u}_j$, \hat{u}_j is the residual of the regression of x_t on D_t and $\hat{\lambda}^2$ is a consistent estimate of the long-run variance of u_t using \hat{u}_t . And, since the test is one-sided right-tailed, the null hypothesis of stationarity is rejected at the $100\alpha\%$ level if the KPSS test statistic K is greater than the $100(1 - \alpha)\%$ quantile from the appropriate asymptotic distribution.

Probability Distributions

Using the generalized Q-Q plot as a guide for determining the maximum domain of attraction of the underlying distribution for the annual maximum rainfall data for Botswana, for the period 1901-2012, we consider two probability distributions in the Gumbel domain (gamma and lognormal) and one in the Frechet domain (BurrXII). The justification for this choice is discussed under the results section of this paper. A description of each of the probability distributions under consideration is contained in Table 1.

Table 1: Descriptions of Probability Distributions used in the Study

Distribution	Probability Density Function	Range	Parameters
Gamma	$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta)$	$0 < x < \infty$	$\alpha > 0$ is a continuous shape parameter; $\beta > 0$ is a continuous scale parameter
Lognormal	$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp(-(\log(x) - \mu)^2 / 2\sigma^2)$	$0 \leq x < \infty$	$-\infty < \mu < \infty$ is the mean and $\sigma > 0$ are the mean and standard deviation of the natural logarithm of X
BurrXII	$f(x) = \frac{\alpha\tau}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} \left\{1 + \left(\frac{x}{\beta}\right)^\alpha\right\}^{-(\tau+1)}$	$0 < x < \infty$	$\beta > 0$ is the scale parameter, $\alpha > 0$ is a continuous shape parameter and $\tau > 0$ is a continuous shape parameter

The most popular maximum likelihood estimation method is used to calculate the parameter estimates for the models under study using various packages of the R language.

Goodness-of-Fit and Model Selection

To assess the fit of the models under consideration, two diagnostic plots, namely, the empirical cumulative distribution function (CDF) and the Quantile plots, and two more formal criteria, the Anderson-Darling (AD) test and Bayesian information criterion (BIC) are used.

The AD test tests the null hypothesis that a sample of size n comes from a population with a specified distribution. It is based on the discrepancy between the empirical cumulative distribution function (ECDF), $F_n(x)$, and the theoretical cumulative distribution function (CDF), $F(x)$, of a specified distribution. This test gives more weight to the tails of the distribution than the more popular Kolmogorov-Smirnov test. As explained -by [19], in the case of fitting maximum values, if $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ denote an increasingly ordered sample of independent observations from a population with distribution function $F(x)$, the Anderson-Darling statistic is calculated using Equation (2.4).

$$A^2 = n/2 - \frac{\sum_{i=1}^n (2i-1) \log(1-z_{n+1-i})}{n} - \sum_{i=1}^n z_i / n \quad (2.4)$$

where $z_i = F_0(x_{(i)})$, the fitted distribution function evaluated at the i th element of the ordered sample of size n . The best-fit model (the model that gives the best fit to the data) is the one with the smallest value of the AD test statistic under consideration.

The BIC is an information theoretic criterion for model selection among a finite set of models with different numbers of parameters. It is given by the formula in Equation (2.5).

$$BIC = -2 \ln f(x|\hat{\theta}) + d \ln n \quad (2.5)$$

where $f(x|\hat{\theta})$ is the score of the maximum likelihood function for the estimated model, n is the sample size and d is the total number of parameters. The model with the smallest value of the BIC is the most plausible.

RESULTS AND DISCUSSIONS

Figure 1 shows a time series plot and histogram of the maximum annual average rainfall for Botswana over the years 1901-2012. The histogram shows that the data have moderate positive skewness.

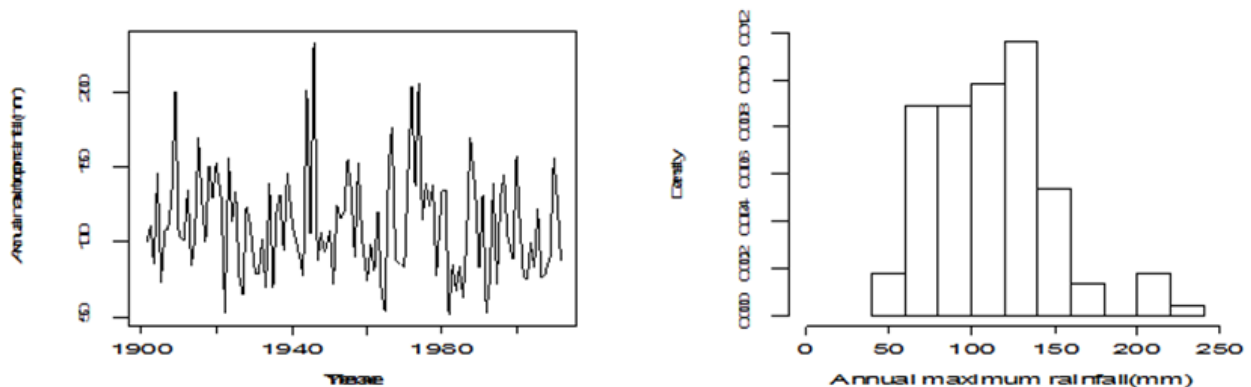


Figure 1: Time Series Plot (Left Panel) and Histogram (Right Panel) of Annual Maximum Rainfall for Botswana, 1901-2012

Descriptive Statistics

Table 2: Descriptive Statistics

N	Mean	Stdev	Q1	Median	Q3	Min	Max	Skew	Kurtosis	Se
112	112.12	12.36	83.45	36.11	134.18	51.02	233.58	0.76	0.55	3.41

The results contained in Table 1, indicate that during the years 1901 to 2012, Botswana received the highest annual maximum rainfall of 233.60 mm and the lowest of 51.02 mm. The value of excess kurtosis of $\gamma_2 = 0.55$ is greater than that of the normal distribution (which is 0), shows that the data are heavy-tailed relative to the normal distribution. And, the coefficient of skewness of $\gamma_1 = 0.76$ indicates that the data are positively skewed, as shown by the histogram in Figure 1. Further to the summary statistics contained in Table 1, a Jarque-Bera test for normality produces a chi-squared value of $\chi^2 = 12.939$ on 2 degrees of freedom, with a *p-value* = 0.00155, indicating that the skewness and kurtosis for these data significantly differ from their expected values under normality.

Testing for Stationarity

The KPSS test gives a *KPSSLevel* = 0.1285, *Truncationlagparameter* = 4, with a *p-value* = 0.1 (Warning message: In *kpsstest* (y): p-value greater than printed p-value). And, since this p-value is significantly large, we fail to reject the null hypothesis of stationarity and conclude that the Botswana annual maximum rainfall data for the stated period are stationary.

Selecting Maximum Domain of Attraction

The generalized Q-Q plot is shown in Figure 2. The plot shows a fairly constant behaviour or trend in the large observations, suggesting the underlying distribution of the data is most probably in the Gumbel domain. This pattern is then followed by an increasing behavior in the largest 4 quantiles, which suggests that a distribution in the Frechet domain might give a reasonably good fit to the annual maximum rainfall for Botswana data. As a result, we focus on probability distributions in the Gumbel domain and, consider two distributions that are widely used for modelling rainfall data: the gamma and lognormal. For comparison, and still based on the behavior of the generalized Q-Q plot in the largest observations, we fit the three-parameter Burr XII, which belongs in the Frechet domain, to the data.

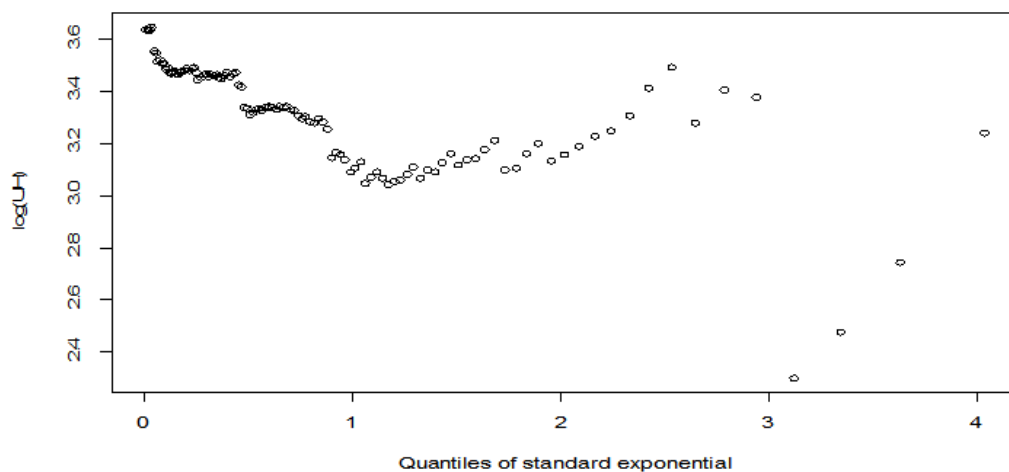


Figure 2: Generalized Q-Q Plot for the Annual Maximum Rainfall for Botswana, 1901-2012

Parameters Estimates

Table 3 contains maximum likelihood estimates (MLEs) of the parameters of the three fitted models for the annual maximum rainfall for Botswana for the stated period. The standard error of each estimate is given in brackets. The standard errors for the gamma and lognormal parameter estimates are generally small, indicating small estimation variance. This is in sharp contrast to the standard errors of the estimates of the BurrXII, which are relatively large, especially in the case of the scale parameter β . This suggests that the BurrXII is the least preferred model of the three models considered for the Botswana annual maximum rainfall data recorded during the period 1901-2012. The next section discusses the goodness-of-fit of the three models.

Table 3: MLE Parameter Estimates of the Fitted Models

Distribution	Estimate (Standard Error)
Gamma(2P)	$\alpha = 10.13566663$ (1.33061037) $\beta = 0.09040273$ (0.01216516)
Lognormal (2P)	$\mu = \text{meanlnx} = 4.6694692$ (0.03001591) $\sigma = \text{Sdlnx} = 0.3176586$ (0.02122351)
BurrXII (3P)	$\alpha = \text{shape1} = 1.459122$ (0.7112521) $\tau = \text{shape2} = 4.835385$ (0.7298173) $\beta = \text{Scale} = 119.664023$ (18.5060675)

Goodness-of-Fit and Model Selection

The Empirical versus theoretical distribution functions (CDF) plot (left) and the Q-Q plot (right) for the gamma, lognormal and Burr XII models are shown in Figure 3

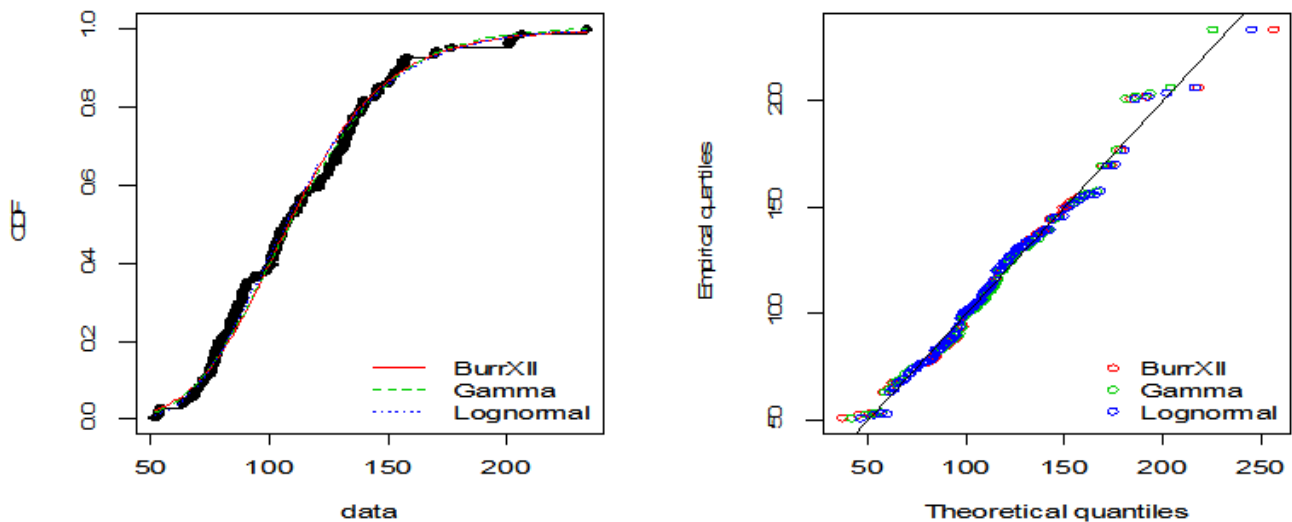


Figure 1: Empirical Versus Theoretical Distribution Functions (Left) and Q-Q Plots for the Models

From the CDF plot, it appears that the three models all fit the data very well. However, the Q-Q plot makes things a bit clearer and indicates that the gamma and the lognormal are very competitive best-fit models, with the BurrXII ranking a distant third.

The findings based on the diagnostic plots in figure 3 are confirmed by the results of the AD test and the BIC contained in Table 2. Both the AD test and the BIC agree on the ranking of the three models: the lognormal distribution

gives the best fit to the data followed closely by the gamma whilst the BurrXII distribution comes a distant third. It must be noted, however, that based on both the diagnostic plots in Figure 3 and goodness-of-fit criteria in Table 4, the BurrXII also gives a reasonably good fit to the annual maximum rainfall data under consideration. Overall, these results are to be expected given the tail behaviour of these data as indicated by the generalized Q-Q plot in Section 3.3 of this paper.

Table 4: The A-D Test Statistics and BIC Scores for the Fitted Models

Goodness-of-Fit Statistic/Criterion	BurrXII	Gamma	Lognormal
Anderson-Darling	0.54129838	0.42747640	0.40346494
Rank	3	2	1
Bayesian Information Criterion	1124.860	1117.472	1116.362
Rank	3	2	1

CONCLUSIONS

In this paper, annual maximum rainfall data for Botswana for the period January 1901 to December 2012 are modelled using the gamma, lognormal and BurrXII probability distribution. Application of the Kwiatkowski-Philips-Schmidt-Shin test of stationarity on the data shows that the time series is stationary. The maximum likelihood method is used to estimate the parameters of each distribution under study. The CDF probability plot and Q-Q plot show that the three models are highly competitive in terms of goodness-of-fit. Moreover, both the AD test and the BIC agree on the ranking of the three models: the lognormal distribution gives the best fit to the data followed closely by the gamma whilst the BurrXII distribution comes a distant third.

REFERENCES

1. Wen, X., Fang, G., Qi, H., Zhou, L. and Gao, Y. (2015). *Changes of temperature and precipitation extremes in China: past and future. Theoretical and Applied Climatology. Vol.10.1007/s00704-015-1584-x*, 369-383.
2. Thuto Mothupi, Wilson Moseki Thupeng, Baitshephi Mashabe and Botho Mokoto (2016). *Estimating Extreme Quantiles of the Maximum Surface Air Temperatures for the Sir Seretse Khama International Airport Using the Generalized Extreme Value Distribution. American Journal of Theoretical and Applied Statistics. Vol. 5, No. 6, 2016, pp. 365-375.*
3. Carter, I. (2007). *Footsteps. Climate Change and Agriculture. Tearfund, UK, pp. 1-4.*
4. Parida, B.P. and Moalafhi D.B. (2008), *Regional rainfall frequency analysis for Botswana using L-Moments and radial basis function network. Physics and Chemistry of the Earth, Parts A/B/C. Vol.33 (8), 614-620.*
5. EM-DAT, 2007. *International Disaster Database. Universite Catholique de Louvain, Brussels, Belgium.*
6. UNDP Botswana, (2012). *Botswana Agriculture Sector Policy Brief 2012. Agriculture and Food Security Policy Brief. Reflecting on the Challenges of Attaining a Green Economy for Botswana.*
7. Chikobvu, D. and Chifurira, R. (2015). *Modelling of extreme minimum rainfall using generalised extreme value distribution for Zimbabwe. SAIr J Sci. Vol.III (9/10), Art. #2014-0271, 8 pages. <http://dx.doi.org/10.17159/sajs.2015/20140271>.*
8. Alam, M.A.; Emura, K.; Farnham, C.; Yuan, J. (2018). *Best-Fit Probability Distributions and Return Periods for Maximum Monthly Rainfall in Bangladesh. Climate 2018. Vol.6, Issue 1, Article NO. 9.*

9. Nadarajah S, Choi D (2007). Maximum daily rainfall in South Korea. *J Earth Sys Sci.* Vol.116 (4):311-320. <http://dx.doi.org/10.1007/s12040-007-0028-0>.
10. Koutsoyiannis D (2004). Statistics of extremes and estimation of extreme rainfall: 11 Empirical investigations of long rainfall records. *Hydrol Sci J.* Vol.49 (4):591-610. <http://dx.doi.org/10.1623/hysj.49.4.591.54424>.
11. Koutsoyiannis D, Baloutsos G (2000). Analysis of a long record of annual maximum rainfall in Athens, Greece and design rainfall inferences. *Nat Hazards.* Vol. 22:29-48. <http://dx.doi.org/10.1023/A:1008001312219>.
12. Crisci A, Gozzini B, Meneguzzo F, Pagliara S and Maracchi G (2002). Extreme rainfall in a changing climate: Regional analysis and hydrological implications in Tuscany. *Hydrol Process.* Vol. 6:1261-1274. <http://dx.doi.org/10.1002/hyp.1061>.
13. Khapalova, E.A., Jandhyala, V.K. and Fotopoulos, S.B. (2013). Change-Point Analysis of Annual Mean Precipitation for Northern, Tropical and Southern Latitude of the Globe in the Past Century. *Journal of Environmental statistics.* Vol.4, Issue 3: 1-21.
14. Fisher, R.A. and Tippett, L.H.C. (1928). On the estimation of the frequency distributions of the largest or smallest member of a sample. *Proceedings of the Cambridge Philosophical Society* 24, 180-190.
15. Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics* 44, 423-453.
16. Coles, S. (2001). *An introduction to Statistical Modelling of Extreme Values.* Springer-Verlag, London.
17. Penalva, H., Nunes, S, and Neves, M.M. (2016). Extreme Value Analysis – A Brief Overview With Applications To Flow Discharge Rate Data In A Hydrometric Station In North of Portugal. *REVSTAT –Statistical Journal.* Vol. 14, No.2, 193-215.
18. Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. (2004). *Statistics of extremes. Theory and Applications, with contributions from de Waal, D.J. and Ferro, C.* Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester.
19. Bora, D. J., & Borah, M. (2017). Regional analysis of maximum rainfall using L-moment and TL-moment: a comparative case study for the north East India. *Journal of Applied and Natural Science,* 9(4), 2366-2371.
20. Thupeng, W.M. (2016). Use of the Three-parameter Burr XII Distribution for Modelling Ambient Daily Maximum Nitrogen Dioxide Concentrations in the Gaborone Fire Brigade. *American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS).* Vol.26, No. 2, 18-32.